The limitations of the Brinkman–Forchheimer equation in modeling flow in a saturated porous medium and at an interface

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This paper is a critique of the ability of the Brinkman–Forchheimer equation to adequately model flow in a porous medium and at a porous-medium/clear-fluid interface. It is demonstrated that certain terms in the equation as commonly used require modification, and that there is a difficulty when using this equation to deal with a stress boundary condition.

Keywords: porous media; interface; Brinkman; Forchheimer, non-Darcy flow

Introduction

This article is a response to the presentation by Vafai and Kim[1] of an "exact solution" to an "important and classical problem" involving the fluid mechanics of the interface region between a porous medium and a fluid layer. The opportunity is taken to comment on the wider aspects of modeling the flow in a saturated porous medium by means of a Brinkman–Forchheimer equation, which is currently popular among contributors to the heat transfer literature.

Vafai and Lim[1] dealt with a steady-state situation, so in order to widen the discussion we will consider a general equation as formulated by Hsu and Cheng,[2] namely,

$$\rho_f\left[\frac{\partial \mathbf{v}}{\partial t}+\bar{\nabla}\cdot\left(\frac{\mathbf{v}\cdot\mathbf{v}}{\varphi}\right)\right]=-\bar{\nabla}p+\mu_f\bar{\nabla}^2\mathbf{v}-\left[\frac{\mu_f\varphi\mathbf{v}}{K}+\rho_f\frac{F\varphi|\mathbf{v}|\mathbf{v}}{K^{1/2}}\right] \tag{1}$$

Here $\mathbf{v}$ is the volume-averaged Darcy seepage velocity, p is the volume-averaged pressure, and φ is the porosity. When use has been made of the incompressibility condition $\bar{\nabla}\cdot\mathbf{v}=0$, Equation 1 can be written as

$$\rho_f\left[\frac{1}{\varphi}\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{\varphi^2}(\mathbf{v}\cdot\bar{\nabla})\mathbf{v}\right]=-\bar{\nabla}p_f+\frac{\mu_f}{\varphi}\bar{\nabla}^2\mathbf{v}-\frac{\mu_f\mathbf{v}}{K}-\frac{F\rho|\mathbf{v}|\mathbf{v}}{K^{1/2}} \tag{2}$$

This may be compared with Darcy's law, which is simply

$$0=-\bar{\nabla}p_f-\frac{\mu_f\mathbf{v}}{K} \tag{3}$$

The last term in Equation 2 is the Forchheimer quadratic drag team in the form recommended by Joseph, Nield, and Papanicolaou.[3] Later in this article, we shall consider the appropriateness of each of the other terms in Equation 2. An equation similar to Equation 2 was obtained by Vafai and Tien.[4]

Status of the Brinkman equation

Equation 1 was obtained by Hsu and Cheng[2] by the process of volume averaging of the momentum equation for flow over a dilute random array of spheres. Our concern is not about the way in which Hsu and Cheng[2] have handled this equation, but rather its use by other authors who have been unaware of its limitations. Equation 1 (or one of its variants) has been popular with authors because, when put in nondimensional form, it has provided them with a parameter—namely, the Darcy number $\mathrm{Da}=K/L^2$, where L is an appropriate macroscopic length scale—that can be used to relate flow in a porous medium (finite Da) with flow in a clear fluid ($\mathrm{Da}\to\infty$). Equation 1 also enables authors to treat a composite region, which is filled partly with porous medium and partly with clear fluid, as a single continuum with certain parameters that change value as one crosses the porous-medium/clear-fluid interface; this facilitates numerical computation. Our point is that this global treatment may fail to deal adequately with the distinctive features of flow in a physical porous medium.

A basic drawback of the Brinkman equation is that its usage cannot be rigorously justified except when the porosity is close to unity. According to the analysis of Lundgren,[5] the self-consistent formulation of Brinkman breaks down when $\varphi \le 0.6$ (which is the case for most naturally occurring media; some exceptions are sponges and some forms of lava). A question arises about the value of the coefficient of $\nabla^2\mathbf{v}$ in Equation 2. Is this effective viscosity μ_e equal to μ_f/φ or something closer to μ_f, as some authors have suggested? It has not been possible to directly check the alternatives against experiments because, as pointed out by Kim and Russel,[6] all the available experimental data pertain to media whose porosity is outside the range for which rigorous theories are valid.

Some authors have justified their use of the Brinkman equation on the grounds that it enabled them to satisfy the no-slip condition on a rigid boundary, a requirement they deemed to be necessary. In most practical cases, the Darcy number Da will be very small and the Brinkman term (the one involving $\nabla^2\mathbf{v}$) will have an effect only in thin layers adjacent to a rigid boundary, specifically within a distance $K^{1/2}$ (dimensional) or $\mathrm{Da}^{1/2}$ (nondimensional) of the boundary. This fact was noted by Vafai and Tien[4] and others. In many cases the reduction in velocity in this thin layer will be masked by an increase in velocity (the channeling effect) due to the increase in porosity near the wall, this increase resulting from the fact that solid particles cannot pack as tightly there as they can in the interior of the medium.[7] If one is not prepared to account for this porosity variation, then the use of the Brinkman term may be a complication that leads to no benefit.

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Modeling an interface

Now we turn to the use of the Brinkman equation in modeling a porous-medium/clear-fluid interface, as in the paper by Vafai and Kim.[1] Since both the Brinkman and Navier–Stokes equations are of second order in spatial derivatives, four matching conditions are needed. On physical grounds, one clearly needs the continuity of four quantities: tangential velocity, normal velocity, tangential stress, and normal stress. There is no problem about implementing the requirement that the velocity components be continuous. One can simply match the seepage velocity in the medium with the velocity in the clear fluid. The interface of the porous medium contains both pores and solid particles. In the pores, the fluid velocity in the porous medium matches with the fluid velocity outside the medium. Over the solid portion of the interface, the velocity is zero both in the medium (obviously) and in the neighboring clear fluid (because of the no-slip condition). The average velocity in the porous medium thus matches with the average velocity in the neighboring clear fluid.

In the case of the tangential stress, the situation is different. One would like to match up the average viscous tangential stress, and hence the average velocity shear, as Vafai and Kim[1] did in their Equation 3c, but there is a problem in doing this that hitherto has been overlooked. Over the pore section of the interface, the velocity shear is continuous; however, this is not the case over the solid section. In the solid the velocity shear is identically zero, but in the adjacent clear fluid it has in general some indeterminate nonzero value. Therefore, the averaged velocity shears do not match. We conclude that Vafai and Kim,[1] following several of their predecessors, have used boundary conditions that overdetermine the physical problems. When one uses the Darcy equation (instead of the Brinkman equation) in the porous medium, the difficulty can be sidestepped. Now one requires only three matching conditions; two of these are provided by the continuity of tangential velocity and normal velocity, and the Beavers–Joseph boundary condition provides the third. This last condition contains an empirical constant, to be determined experimentally, and this permits the needed flexibility in modeling the tangential stress requirement.

Vafai and Kim[1] wisely stated that they were "not trying to resolve a complex question with regard to the physical nature of the interface." We now emphasize this. An "exact" solution of the Brinkman–Forchheimer/Navier–Stokes equation system is welcome in the same way that an "exact" similarity solution of a boundary-layer system of equations is welcome, but its limitations should be appreciated. The use of the Brinkman equation leads to an overestimate of the extent to which motion produced in the clear fluid will penetrate into the porous medium.

The situation with respect to the averaging of normal stress is somewhat similar to that of tangential stress, but there is an additional factor involved. The normal stress is the sum of a pressure term and a viscous stress term. Some authors, e.g., Vafai and Kim,[8] have argued that the pressure, being an intrinsic quantity, has to be continuous across the interface. Since the total normal stress is continuous, that means that the viscous component of stress, and hence the normal rate of strain, must also be continuous. Such authors have overdetermined the system of equations. It is true that the pressure has to be continuous on the microscopic scale, but that does not mean that it has to be continuous on the macroscopic scale. The interface surface is an idealization of a thin layer in which the pressure can change substantially because of the presence of the solid material; the pressure on one side of a solid particle can differ from the pressure on the other side. In practice, the viscous term in the normal stress may be small compared with the pressure, and in this case the continuity of total normal stress does reduce to the approximate continuity of pressure.

The local time-derivative inertial term

We now consider the inertial terms on the left-hand side of Equation 2. It has long been realized that the local time-derivative inertial term $\varphi^{-1}\,\partial\mathbf{v}/\partial t$ is usually small compared with the Darcy drag term on the right-hand side and so can be neglected. In most practical situations the velocity responds to an imposed pressure change within a second or less. (Incidentally, it is not necessary to retain the $\varphi^{-1}\,\partial\mathbf{v}/\partial t$ term in the investigation of oscillatory convective instabilities; these involve a beating between two or more convective modes and so differ from oscillatory hydrodynamic instabilities.) The $\varphi^{-1}\,\partial\mathbf{v}/\partial t$ term may be important if an oscillatory pressure gradient is imposed. The question then arises as to whether the coefficient φ^{-1} is correct or not. Hitherto it has been assumed that the coefficient is always correct. It has not been appreciated that when the pores are organized on a scale greater than the characteristic pore diameter, e.g., when the medium contains tubes or channels, the assumption that a partial derivative with respect to time permutes with a volume average then breaks down, and therefore the $\varphi^{-1}\,\partial\mathbf{v}/\partial t$ term is incorrect. The response to pressure changes in wide tubes is slower than that in narrow tubes.

To illustrate this, let us consider an ideal medium, one in which the pores are identical parallel tubes of uniform circular cross section of radius a. (One can make the structure connected

Notation

a	Tube radius, m
$\mathbf{c}_a$	Acceleration coefficient tensor
Da	Darcy number, K/L^2
D_p	Particle/pore diameter, m
F	Forchheimer coefficient
J_0	Bessel function
K	Permeability, m^2
L	Characteristic length, m
N_{ij}	Contribution from fractures to the acceleration tensor $(c_a)_{ij}$
p	Volume-averaged fluid pressure, Pa
p_f	Intrinsic fluid pressure, Pa
t	Time, s
t_0	Characteristic time, s
$\mathbf{v}$	Darcy seepage velocity, m s^{-1}
$\mathbf{V}$	Intrinsic velocity, $\mathbf{v}/\phi$

Greek symbols

δ_{ij}	Kronecker delta tensor
λ_1	Zero of the Bessel function J_0
μ_f	Dynamic viscosity of fluid, kg m^{-1} s^{-1}
ρ_f	Density of fluid, kg m^{-3}
φ	Porosity

by adding some very fine connecting tubes without qualitatively affecting the argument.) The time-dependent Darcy equation

$$\rho_f \varphi^{-1}\, \partial \mathbf{v}/\partial t = -\nabla p - (\mu/K)\mathbf{v} \tag{4}$$

leads to the prediction that in the presence of a constant pressure gradient, any transient will decay like $\exp[-(\mu\varphi/K\rho_f)t]$. From the exact solution for a circular tube (see, for example, formula (4.3.19) of Batchelor[9]), one concludes that a transient will decay approximately like $\exp[-(\lambda_1^2\mu/a^2\rho_f)t]$, where $\lambda_1 = 2.405$ is the smallest positive root of $J_0(\lambda) = 0$, where J_0 is the Bessel function of the first kind of order zero. In general, these two exponential decay terms will not be the same. It appears that the best that one can do is to replace Equation 4 by

$$\rho_f \mathbf{c}_a \cdot (\partial \mathbf{v}/\partial t) = -\nabla p - (\mu/K)\mathbf{v} \tag{5}$$

where $\mathbf{c}_a$ is a constant tensor that depends sensitively on the geometry of the porous medium and that is determined mainly by the nature of pore tubes of largest cross sections (since in the narrower ones, the transients decay more rapidly). We propose that $\mathbf{c}_a$ be called the acceleration coefficient tensor of the porous medium. In general it will take the form

$$(\mathbf{c}_a)_{ij} = \varphi^{-1}\delta_{ij} + N_{ij} \tag{6}$$

where N_{ij} is the contribution from "fractures." For the special medium introduced above, in which we have unidirectional flow, the acceleration coefficient will be a scalar,

$$c_a = a^2/\lambda_1^2 K \tag{7}$$

If the permeability K is estimated by means of the Carman–Kozeny formula

$$K = \frac{D_p^2}{180}\frac{\varphi^3}{(1-\varphi)^2} \tag{8}$$

and if the pore/particle diameter D_p can be identified with a/γ where γ is some constant, then

$$c_a = 180\gamma^2(1-\varphi)^2/\lambda_1^2\varphi^3 = 31.1\gamma^2(1-\varphi)^2/\varphi^3 \tag{9}$$

The ratio of the time-derivative term to the Darcy resistance term is $c_a\rho_f K/\mu t_0$, where t_0 is the characteristic time of the process being investigated. Transients decay rapidly when this quantity is small, as is usually the case. The exception is when the kinematic viscosity μ/ρ_f of the fluid is small compared with K/t_0.

The convective inertial term

Finally, we consider the validity of the inclusion of the second term on the left-hand side of the Brinkman–Forchheimer equation (Equation 2). Joseph, Nield, and Papanicolaou[3] expressed the view that the convective term involving $(\mathbf{v}\cdot\bar{\nabla})\mathbf{v}$ in a Forchheimer equation should be omitted because the effect of inertial terms quadratic in the velocity is already properly accounted for by the quadratic drag term involving $|\mathbf{v}|\mathbf{v}$. We now give a new argument for why it is inappropriate to include the $(\mathbf{v}\cdot\bar{\nabla})\mathbf{v}$ term. If it is included, and if the forces represented by the right-hand side of Equation 2 are in balance, then that equation reduces to

$$\frac{D}{Dt}(\rho_f \mathbf{v}) = 0 \tag{10}$$

where D/D_t is the material derivative $\partial/\partial t + (\mathbf{v}\cdot\bar{\nabla})$. Equation 10 implies that a small particle retains its momentum when it is displaced to an arbitrary neighboring point. This is true for the case of a clear fluid, but in general it is incorrect for a porous medium, because solid material will have an impeding effect. (We are, of course, considering a fixed solid matrix.)

To further illustrate this objection, we consider the extent to which it is possible to transmit longitudinal momentum in a transverse direction. To highlight the argument, let us consider a medium in which the pores consist of channels along the x-, y-, and z-directions. If one forces fluid to flow down a single x-channel, that will cause flow along the y- and z-channels but will not produce any significant flow *on the average* in neighboring x-channels. A consequence is that, on physical grounds, one would expect that it should be difficult to produce significant motion in the bulk of a dense porous medium, with a fixed solid matrix, by moving just a rigid boundary; one would expect significant motion to be confined to a thin layer near the boundary. Indeed, that is the form of motion predicted when one solves a Brinkman equation with the $(\mathbf{v}\cdot\bar{\nabla})\mathbf{v}$ term omitted (see, for example, Nield[10]), but it is not the form predicted if one includes the $(\mathbf{v}\cdot\bar{\nabla})\mathbf{v}$ term. In the latter case, the prediction is that all the fluid will ultimately be set in motion. (A further consequence of our physical argument is that true turbulence, in which there is a cascade of energy from large eddies to smaller eddies, does not occur on a macroscopic scale in a dense porous medium.)

This leaves the question of why the averaging process leads to misleading results. The averaging process leads to the loss of vital information about the way in which the geometry of the solid matrix affects the flow. It has been assumed by some authors, e.g., Vafai and Tien,[4] that it is sufficient to allow for this loss of information by *adding* a Forchheimer quadratic drag term. It is now suggested that a better approximation for the case of dense porous media is obtained by *substituting* a quadratic drag term for the $(\mathbf{v}\cdot\bar{\nabla})\mathbf{v}$ term. As Joseph, Nield, and Papanicolaou[3] pointed out, this macroscopic drag arises as a result of form drag on the solid particles. It is independent of the boundary friction (and hence independent of the viscosity). It acts in a direction opposite to the seepage velocity $\mathbf{v}$. It has been shown[11–13] that, while microscopic inertial forces may be important, macroscopic inertial forces are negligible in comparison with the macroscopic drag forces in the range of pore Reynolds numbers encountered in most practical situations. It seems that the necessity for the fluid to flow around the solid particles gives rise to a reduction in the coherence of the fluid momentum pattern, so that on the macroscopic scale there is negligible net transfer of momentum in a direction transverse to the seepage velocity vector.

As a footnote, it is pointed out that there is a further peculiarity arising from the approach of Vafai and Tien.[4] They gave a demonstration leading to the conclusion that "a formulation like Brinkman's is applicable only for two-dimensional problems." On general physical grounds, one would expect that the applicability (or lack of it) would be independent of whether the problem was two- or three-dimensional.

Conclusion

Our conclusion is that flow in a dense porous medium (one in which the porosity is not close to unity) is best modeled by omitting the $(\mathbf{v}\cdot\bar{\nabla})\mathbf{v}$ term and modifying the time-dependent term and the Brinkman term so that the momentum equation is

$$\rho_f \mathbf{c}_a \cdot \frac{\partial \mathbf{v}}{\partial t} = -\bar{\nabla} p_f + \mu_{\text{eff}}\bar{\nabla}^2\mathbf{v} - \frac{\mu_f \mathbf{v}}{K} - \frac{F\rho|\mathbf{v}|\mathbf{v}}{K^{1/2}} \tag{11}$$

Further, if in that case (for a dense medium) one has to include the Brinkman term, then porosity variation should also be allowed for. A porous-medium/clear-fluid interface is best dealt with by dropping the Brinkman term and using the Beavers–Joseph boundary condition.

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